Quintessence inhomogeneous cosmology

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Abstract

The Einstein–Klein–Gordon field equations are solved in a inhomogeneous shear–free universe containing a material fluid, a self–interacting scalar field, a variable cosmological term, and a heat flux. A quintessence–dominated scenario arises with a power–law accelerated expansion compatible with the currently observed homogeneous universe.

1 Introduction

Recently, there have been claims in the literature that the Universe, besides its content in normal matter and radiation, must possess a not yet identified component (usually called *quintessence* matter, Q-matter for short) [1], [2], [3], characterized by a negative pressure, and possibly a cosmological term. These claims were prompted at the realization that the clustered matter component can account at most for one third of the critical density. Therefore, an additional "soft" (i.e. non-clustered) component is needed if the critical density predicted by many inflationary models is to be achieved.

Very often the geometry of the proposed models is very simple, just Friedmann-Lemaître-Robertson-Walker (FLRW). In constrast to FLRW models, inhomogeneous spaces are in general compatible with heat fluxes, and these might imply important consequences such as inflation or the avoidance of the initial singularity [4]. Here we focus on an isotropic but inhomogeneous spherically symmetric universe which besides a material fluid contains a self-interacting scalar field (which can be interpreted as Q-matter), and a cosmological term, Λ which, in general, may vary with time.

Density inhomogenities triggered by gravitational instability must be present at any stage of evolution. We only mention that the negative pressure associated to Q-matter and Λ will tend to slow down the growing modes (see *e.g.* [1], [5]), and shift the epoch of matter-radiation equality toward more recent times.

2 Einstein-Klein-Gordon field equations

Let us consider a shear–free spherically–symmetric spacetime with metric [6]

$$ds^2 = \frac{1}{F(t,\, r)^2}\left[-v(t,r)^2 \;\; d\,t^2 \;+\; d\,r^2 + r^2\,\mathrm{d}\Omega^2\right]. \tag{1}$$

where as usual $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. Units have been chosen so that $c = G = 1$. As sources of the gravitational field we take: a fluid of material energy density $\rho_f = \rho_f(r,t)$, hydrostatic pressure $p_f = p_f(r,t)$, with a radial heat flow ($q_r = q_r(r,t)$ and $q_t = q_\theta = q_\phi = 0$), plus a cosmological term, related to the energy density of vacuum by $\Lambda = 8\pi\rho_{vac}$, that depends only on time $\Lambda = \Lambda(t)$, and a self-interacting scalar field ϕ driven by the potential $V(\phi)$ whose equation of state is $p_\phi = (\gamma_\phi - 1)\,\rho_\phi$. Hence the scalar field can be interpreted as Q-matter -see e.g. [3]. The stress energy-tensor of the normal matter, with a heat flow, plus Q-matter (scalar field) and the cosmological term is

$$T^i_k = (\rho_f + \rho_\phi + p_f + p_\phi)u^i u_k + (\Lambda - p_f - p_\phi)\delta^i_k + q^i u_k + q_k u^i\,, \tag{2}$$

As equation of state for the fluid we choose $p_f = (\gamma_f - 1)\,\rho_f$ where γ_f is a function of t and r. Taking into account the additivity of the stress-energy tensor it makes sense to consider an effective perfect fluid description with equation of state $p = (\gamma - 1)\,\rho$ where $p = p_f + p_\phi$, $\rho = \rho_f + \rho_\phi$ and

$$\gamma = \frac{\gamma_f\rho_f + \gamma_\phi\rho_\phi}{\rho_f + \rho_\phi}\,, \tag{3}$$

is the overall (i.e. effective) adiabatic index. The requirement that the cosmological term Λ is just a function of t leads to the restriction that γ also depends only on t to render the system of Einstein equations integrable. The nice result we are seeking is a solution that has an asymptotic FLRW

stage, with Λ evolving towards a constant, and the heat flow vanishing in that limit [7].

To write the Einstein equations we use the ansatz $F = a(t) + b(t)x$ and $v = c(t) + d(t)x$ with the constraints $a(t)\,d(t) = b(t)\,c(t)$. This set of metrics contains those of Modak ($b = 0$) [8], Bergmann ($c = a, d = b$) [9] and Maiti ($b = d = k\,a/4$, with $k = 0, \pm 1$) [10]. Another possibility arises when $d = 0$, then re-defining the time by $vdt \to dt$, the Einstein equations are

$$\rho + \Lambda = 12ab + 3\dot{a}^2 + 6\dot{a}\dot{b}x + 3\dot{b}^2 x^2, \tag{4}$$

$$p - \Lambda = \left(2b\ddot{b} - 3\dot{b}^2\right) x^2 + 2\left(2b^2 - 3\dot{a}\dot{b} + a\ddot{b} + \ddot{a}b\right) x - 8ab - 3\dot{a}^2 + 2a\ddot{a}\,, \tag{5}$$

$$q_r = -4\sqrt{x}\dot{b}\,(a + bx)^2\,, \tag{6}$$

where $x = r^2$ and the overdot indicates $\partial/\partial t$. Imposing that $\Lambda = \Lambda(t)$, the general solution to these equations has the form

$$a = -2\exp\left(\int dt\, w\right) \int dt\, w^2 \int \frac{dt}{w^2}\,, \qquad b = \exp\left(\int dt\, w\right), \tag{7}$$

where $w = 2/\int dt(2 - 3\gamma)$, provided $\gamma \neq 2/3$. Inserting (7) in (4), (5) and (6) we easily compute the cosmological constant and the heat flow. Finally the FLRW metric is

$$ds^2 = \frac{1}{(1 + Mr^2)^2}\left[-d\tau^2 + R^2\left(dr^2 + r^2\, d\Omega^2\right)\right]\,, \tag{8}$$

where we have introduced the time coordinate $d\tau = dt/a$, $M = b/a$ and $R = 1/|a|$. This metric is conformal to FLRW, and the conformal factor approaches unity when $M \to 0$.

3 Constant adiabatic index

When γ is a constant different from 2/3, the general solution of (4) and (5) becomes

$$a(t) = C_1 \Delta t^{-\frac{2}{3\gamma - 2}} + C_2 \Delta t^{-\frac{3\gamma}{3\gamma-2}} - \frac{1}{3} K \Delta t^{6\frac{\gamma-1}{3\gamma-2}}\,, \tag{9}$$

$$M(t) = K\left(C_1 + C_2 \Delta t^{-1} - \frac{1}{3} K \Delta t^2\right)^{-1}. \tag{10}$$

Two alternatives of asymptotically expanding universes appear depending on the map between t and τ.

Case $\Delta t \to 0$

In this limit we obtain

$$R(\tau) \simeq \frac{1}{\mid C_2 \mid}\left[\frac{2C_2\,(1-3\gamma)}{2-3\gamma}\Delta\tau\right]^{\frac{3\gamma}{2(3\gamma-1)}}, \tag{11}$$

$$\Lambda(\tau) \simeq \frac{3\,(2-3\gamma)^2}{4\,(1-3\gamma)^2\,\Delta\tau^2}, \tag{12}$$

$$q_r(r,\tau) \simeq \frac{8KC_2}{3\gamma-2}\, r\left[\frac{2C_2\,(1-3\gamma)}{2-3\gamma}\Delta\tau\right]^{\frac{9\gamma}{2(1-3\gamma)}}, \tag{13}$$

for $\gamma \neq 1/3$. When $1/3 < \gamma < 2/3$ we have, for large cosmological time τ, an accelerating universe that homogenizes with vanishing cosmological term and heat flow. In this stage we have a final power-law expansion era. For $\gamma = 1/3$ we have asymptotically a de Sitter universe with finite limit cosmological term. For the remaining values of γ the universe begins at a homogeneous singularity with a divergent cosmological term. When $\gamma < 1/3$, the heat flux asymptotically vanishes near the singularity, while for $\gamma > 2/3$ it diverges.

Case $\Delta t \to \infty$

In this limit we obtain

$$R(\tau) \simeq -\frac{3}{K}\left[\frac{K\,(4-3\gamma)}{3\,(2-3\gamma)}\Delta\tau\right]^{\frac{6(1-\gamma)}{4-3\gamma}}, \tag{14}$$

$$\Lambda(\tau) \simeq -\frac{24\,(2-3\gamma)^2}{(4-3\gamma)^2\,\Delta\tau^2}, \tag{15}$$

$$q_r(r,\tau) \simeq -24\frac{(2-3\gamma)^2}{(4-3\gamma)^3}\frac{r}{\Delta\tau^3}, \tag{16}$$

for $\gamma \neq 2/3$. When $2/3 \leq \gamma \leq 1$ the universe homogenizes for large cosmological time with vanishing cosmological term and heat flow. When $\gamma = 1$, the late time evolution changes to an asymptotically Minkowski stage. For $1 < \gamma < 4/3$ the universe starts homogeneously in the remote past with a vanishing scale factor, cosmological term and heat flow. For the remaining values of γ the universe begins at a homogeneous singularity with a divergent cosmological term.

An exact solution with explicit dependence on the asymptotic cosmological time τ can be found when the integration constants C_1 and C_2 vanish. In such a case the metric is

$$ds^2 = \frac{1}{\left(1 + m\Delta\tau^{2\frac{2-3\gamma}{4-3\gamma}} r^2\right)^2} \left[-d\tau^2 + \Delta\tau^{12\frac{1-\gamma}{4-3\gamma}} \left(d\,r^2 + r^2\, d\Omega^2\right)\right], \quad (17)$$

where m is a redefinition of the old integration constant K, the adiabatic index γ and r_0. The last constant was introduced by scaling the radial coordinate $r \to r_0\, r$.

4 Asymptotic evolution to a quintessence–dominated era

As a first stage of towards more general scenarios with a slowly time–varying γ, we will explore a model that evolves towards an asymptotic FLRW regime dominated by Q–matter (i.e. the scalar field). We will show that this system approaches to the constant γ solutions for large times found above. In this regime the equations of Einstein-Klein-Gordon become

$$3H^2 \simeq \rho_f + \frac{1}{2}\dot{\phi}^2 + V(\phi) + \Lambda\,, \quad (18)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} \simeq 0, \quad (19)$$

where $H = \dot{R}/R$ and a dot means $d/d\tau$ in this section. In last section we found that the general asymptotic solution for the scale factor $R(\tau) \propto \Delta\tau^{\alpha}$ has the power–law behaviors (11), (14) for any value of the effective adiabatic index γ. Then, using these expressions and (12) and (15) together with (18) and (19), we can investigate the asymptotic limit in which the energy of

the scalar field dominates over the contribution of the perfect fluid. In the regime that $3\alpha\gamma_f > 2$ the adiabatic scalar field index can be approximated by

$$\gamma_\phi \simeq \frac{2}{3\alpha}\left[1+\left(1-\frac{3\gamma_f}{2}\sigma\right)\right], \tag{20}$$

where $\sigma = \rho_f/\rho_\phi \ll 1$. Inserting these equations in (3) we obtain the first correction to the effective adiabatic index

$$\gamma \simeq \frac{2}{3}\left[1 \pm \sqrt{\frac{11}{12}\left(3\gamma_f - 2\right)\sigma}\right] \tag{21}$$

The negative branch of (21) yields a consistent asymptotic solution for the range $\frac{1}{3} < \gamma < \frac{2}{3}$. We note that this solution describes a deflationary stage with a limiting exponent $\alpha = 1$.

Oftenly power-law evolution of the scale factor is associated with logarithmic dependence of the scalar field on proper time [13]. Thus, assuming that $\phi(\tau) \simeq C \ln \tau$ with the constant C to be determined by the system of equations (18) and (19), and using these expressions together with (12) and (15) in (18) and (19) it follows that the leading term of $V(\phi)$ for large ϕ is

$$V\left(\phi\right) \simeq V_0 e^{-A\phi} \tag{22}$$

Using the dominant value of the effective adiabatic index we find $A^2 = 2$, $V_0 = 2$ and $C = 1/\sqrt{2}$.

The models considered in this section are based on the notion of "late time dominating field" (LTDF), a form of quintessence in which the field ϕ rolls down a potential $V(\phi)$ according to an attractor solution to the equations of motion. The ratio σ of the background fluid to the field energy changes steadily as ϕ proceeds down its path. This is desirable because in that way the Q-matter ultimately dominates the energy density and drives the universe toward an accelerated expansion [11], [12].

5 Concluding remarks

We have investigated a class of solutions of the Einstein field equations with a variable cosmological term, heat flow and a fluid with variable adiabatic index that includes those of Modak, Bergmann and Maiti and contains a new

exact conformally flat solution. We have found that asymptotically expanding universes occur when $1/3 < \gamma < 1$ that homogenizes for large cosmological time with vanishing cosmological term and heat flow. For $1/3 < \gamma < 2/3$ the evolution is given by (11) and corresponds to a power-law accelerated expansion for large cosmological time τ. On the other hand, when $2/3 \leq \gamma < 1$ even though an asymptotic negative cosmological term occurs, the universe evolves toward a decelerated expansion. The particular case $\gamma = 1/3$ leads asymptotically to a de Sitter universe with a finite limit for Λ. We have shown that homogeneization occurs also for a time dependent adiabatic index provided it has a constant limit, for $t \to 0$ and $t \to \infty$, repectively, and is analytic about these points.

We have carried out a detailed analysis of a model in which Q–matter dominates over cold dark matter. This LTDF solution is an attractor because, even for large initial inhomogeneities and a wide range of initial values for ϕ and $\dot{\phi}$, the evolution approaches a common path. It was shown that this model can be realized for a wide range of potentials provided they have an exponential tail. Our LTDF model only requires that the potential has an asymptotic exponential shape for large ϕ. So, the interesting and significant features of our model are: (a) a wide range of initial conditions are drawn towards a common evolution; (b) the LTDF maintain some finite difference in the equation of state such that the field energy eventually dominates and the universe enters a period of acceleration.

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